15

DISSERTATIO ACADEMICA

THEORIAM ÆQUATIONUM FUNCTIONALIUM DUARUM VARIABILIUM EJUSQUE IN DOCTRINA SERIERUM USUM EXHIBENS;

QUAM

CONSENSU AMPLISS. FACULTATIS PHILOSOPH.

AD IMPERIALEM ACAD. ABOËNSEM,

PRÆSIDE

Mag. NATH. G. AF SCHULTÉN,

Mathematum Professore Publ. & Ord.,
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PRO GRADU PHILOSOPHICO

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P. VI.

ABOÆ, Typis FRENCKELLIANIS.

THESES.

I.

Doctrinam proportionum geometricam ea quidem methodo, ut ad *commensurabiles* tantummodo ratio habeatur quantitates, absolvi simplicissime posse, concedendum est. Cum ex altera vero parte negari nequeat, universalitati in Geometria debitæ hunc minime consulere rei conspectum, peccasse revera quotquot istam adeo proposuerint theoriam Auctores Mathematicos, contendere possumus.

II.

Casus vero *incommensurabilium* eo tantum pacto apte methodoque Geometricæ genuinæ convenienter tractari hoc in argumento potest, ut vestigia antiqui illius inclytique Geometriæ patris *Euclidis* quam proxime persequamur.

III.

Quod tamen non ita intellectum volumus, ut omnia quæ ab Eo hanc respicientia materiem proposita sint, probanda commendandaque statuamus.

IV.

Algebra quidem non eam tantum ob causam *Arithmetica Universalis* dicenda est, quod solutionibus ejus ope eruendis infiniti comprehendantur casus particulares Arithmetici.

V.

Demonstrationes sic dictas *indirectas* tanti in Mathesi haberi usus, ut vel directis ipsis potiorem tueantur locum, adserere non dubitamus.

ri vero facillime posse ipsam ψ ope allatæ supra formæ I) perspicuum est, multiplicando scilicet tantum numeratorem atque denominatorem fractionis

$$\frac{g(y-k)^p + h(y-k)^q + \&c.}{y(y+a')(y+b')\ldots(y+k')}$$

per illos factores formæ

$$y + l'$$

F (desig-

$$\Sigma \frac{g(y+k-k)^p + h(y+k-k)^q + \&c.}{(y+k)(y+k+a')(y+k+b')\ldots(y+k+k')} = \psi(y+k),$$

h. e.

$$\Sigma \frac{gy^p + hy^q + \&c.}{(y+k)(y+k+a')(y+k+b')\ldots(x+k+k')} = \psi(y+k);$$

hincque, mutata y in x,

$$\Sigma \frac{gx^p + hx^q + \&c.}{(x+k)(x+k+a')(x+k+b')\ldots(x+k+k')} = \psi(x+k),$$

i. e.

$$\Sigma \frac{gx^p + hx^q + \&c.}{(x+a)(x+b)(x+c)\ldots(x+l)} = \psi(x+k),$$

ut allatum supra est.

(designante l' numerum positivum integrum), quibus opus iste habeat denominator, ut ad faciem completam

$$y\,(y+1)\,(y+2)\,(y+3)\;\ldots\;(y+k'),$$

quæ in I) scilicet forma est denominatoris, reducetur. Sic scilicet mansurum adhuc esse patet fractionis de qua agitur numeratorem sub forma rationali integra, ut et valere quoque memoratam de summa potestate ipsius x in numeratore numeroque factorum in denominatore conditionem, eamdemque igitur fractionem ad formam ipsam in I) obvenientem

$$\frac{gx^p + hx^q + \&c.}{x\,(x+1)\,(x+2)\ldots(x+m)},$$

ubi y tantum pro x posita est, reductam haberi, manifestum est.

Forma autem II) generalior multo reddi potest, observando, quod in genere ad eam revocari queat

$$(gx^p + hx^q + \&c.)\,.\,a^x\varkappa\;\ldots\;\text{IV}),$$

designante $\varkappa$ functionem quamcumque rationalem integram ipsarum

Sin

$$\operatorname{Sin}(\alpha+\beta x),\ \operatorname{Sin}(\alpha'+\beta' x),\ \operatorname{Sin}(\alpha''+\beta'' x),\ \&c.$$
$$\operatorname{Cos}(\gamma+\delta x),\ \operatorname{Cos}(\gamma'+\delta' x),\ \operatorname{Cos}(\gamma''+\delta'' x),\ \&c.$$

quotcumque eædem fuerint, i.e., facto in genere

$$\begin{aligned}\varkappa = \varepsilon &+ \zeta.\operatorname{Sin}(\alpha+\beta x)^{k}.\operatorname{Sin}(\alpha'+\beta' x)^{k'}\ \&c.\ \operatorname{Cos}(\gamma+\delta x)^{l}.\operatorname{Cos}(\gamma'+\delta' x)^{l'}.\ \&c.\\ &+ \zeta_1.\operatorname{Sin}(\alpha_1+\beta_1 x)^{k_1}.\operatorname{Sin}(\alpha_1'+\beta_1' x)^{k_1'}.\&c.\operatorname{Cos}(\gamma_1+\delta_1 x)^{l_1}.\operatorname{Cos}(\gamma_1'+\delta_1' x)^{l_1'}.\&c.\\ &+ \zeta_2.\operatorname{Sin}(\alpha_2+\beta_2 x)^{k_2}.\operatorname{Sin}(\alpha_2'+\beta_2' x)^{k_2'}.\&c.\operatorname{Cos}(\gamma_2+\delta_2 x)^{l_2}.\operatorname{Cos}(\gamma_2'+\delta_2' x)^{l_2'}.\&c.\\ &+ \&c.,\end{aligned}$$

ubi coefficientes scilicet ε, ζ, α, β, α', β', . . γ, δ, γ', δ', . . ζ_1, α_1, β_1, α_1', β_1', . . γ_1, δ_1, γ_1', δ_1', . . &c. ab x non pendentes censendi sunt, ipsique exponentes k, k', . . l, l', . . k_1, k_1', . . l_1, l_1', . . k_2, k_2', . . l_2, l_2', . . &c. numeri integri positivi *).

Quæ

*) Sub generaliori quodammodo adhuc forma ipsam poni posse $\varkappa$ notari convenit, statuendo eam scilicet functionem rationalem integram non tantum *Sinus* et *Cosinus* quantitatum formæ

$$A + Bx$$

sed etiam ejusmodi quantitatum *Sinus versi* atque *Cosinus versi*, cum hujus scilicet generis functio ad allatam supra ipsius $\varkappa$ formam valorum ope

$$\operatorname{Sin\,vers} u = 1 - \operatorname{Cos} u$$
$$\operatorname{Cos\,vers} u = 1 - \operatorname{Sin} u,$$

facillimi revocari queat.

Quæ quidem reductio eo fiet pacto, ut functio ipsa x in seriem vertatur formæ

$$d \operatorname{Sin}(e+fx) + d' \operatorname{Sin}(e'+f'x) + d'' \operatorname{Sin}(e''+f''x) + \&c.,$$

coefficientibus $d, e, f, d', e', f', d'', e'', f''$, &c. ab x non pendentibus. Fieri autem id poterit formularum ope notissimarum, scilicet:

$$1{:}o \quad \operatorname{Cos} u^n = \frac{1}{2^{n-1}} \Big(\operatorname{Cos} nu + n \operatorname{Cos}(n-2)u + \frac{n(n-1)}{1.2} \operatorname{Cos}(n-4)u$$

$$+ \frac{n(n-1)(n-2)}{1.2.3} \operatorname{Cos}(n-6)u + \&c.\Big),$$

ubi omittendi scilicet sunt termini omnes Cosinus Arcuum negativorum involventes, nec non, in casu quo n numerus est *par*, ultimi termini (Cos o continentis) dimidius tantum adhibendus est coefficiens; nec non

2:o Quando n numerus est *par*

$$\operatorname{Sin} u^n = \pm \frac{1}{2^{n-1}} \Big(\operatorname{Cos} nu - n \operatorname{Cos}(n-2)u + \frac{n(n-1)}{1.2} \operatorname{Cos}(n-4)u$$

$$+ \frac{n(n-1)(n-2)}{1.2.3} \operatorname{Cos}(n-6)u - \&c.\Big),$$

ubi signum + valet si per 4 divisibilis sit n, — vero si tantummodo per 2, ipsaque ceterum series

ries eodem prorsus modo ac allata nuperrime adhibenda est; atque

3:o Quando n est *impar*,

$$\text{Sin}\, u^n = \pm \frac{1}{2^{n-1}}\Big(\text{Sin}\, nu - n\, \text{Sin}\,(n-2)u + \frac{n(n-1)}{1\,.\,2}\, \text{Sin}\,(n-4)u$$

$$- \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\, \text{Sin}\,(n-6)u + \&\text{c.}\Big),$$

ubi $+$ adhibendum est, si per 4 divisibilis sit $n-1$, $-$ vero si tantum per 2, terminique omnes arcus negativos complectentes, uti antea, omittendi sunt *): tandemque

4:o

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*) Positis in allatis nuper tribus formulis successive $n = 1$, 2, 3, 4, &c. tabellæ prodeunt sequentes, quas computatas habere commodum est:

$\text{Cos}\, u = \text{Cos}\, u$

$\text{Cos}\, u^2 = \frac{1}{2}(\text{Cos}\,2u + 1)$

$\text{Cos}\, u^3 = \frac{1}{4}(\text{Cos}\,3u + 3\text{Cos}\,u)$

$\text{Cos}\, u^4 = \frac{1}{8}(\text{Cos}\,4u + 4\text{Cos}\,2u + 3)$

$\text{Cos}\, u^5 = \frac{1}{16}(\text{Cos}\,5u + 5\text{Cos}\,3u + 10\text{Cos}\,u)$

$\text{Cos}\, u^6 = \frac{1}{32}(\text{Cos}\,6u + 6\text{Cos}\,4u + 15\text{Cos}\,2u + 10)$

4:o

$$\operatorname{Sin} u \operatorname{Cos} v = \tfrac{1}{2} (\operatorname{Sin}(u+v) + \operatorname{Sin}(u-v))$$

$$\operatorname{Sin} u \operatorname{Sin} v = \tfrac{1}{2} (\operatorname{Cos}(u-v) - \operatorname{Cos}(u+v))$$

Cos$u$

$$\operatorname{Cos} u^7 = \tfrac{1}{64} (\operatorname{Cos} 7u + 7\operatorname{Cos} 5u + 21\operatorname{Cos} 3u + 35\operatorname{Cos} u)$$

$$\operatorname{Cos} u^8 = \tfrac{1}{128} (\operatorname{Cos} 8u + 8\operatorname{Cos} 6u + 28\operatorname{Cos} 4u + 56\operatorname{Cos} 2u + 35)$$

$$\operatorname{Cos} u^9 = \tfrac{1}{256} (\operatorname{Cos} 9u + 9\operatorname{Cos} 7u + 36\operatorname{Cos} 5u + 84\operatorname{Cos} 3u + 126 \operatorname{Cos} u)$$

$$\operatorname{Cos} u^{10} = \tfrac{1}{512} (\operatorname{Cos} 10u + 10\operatorname{Cos} 8u + 45\operatorname{Cos} 6u + 120\operatorname{Cos} 4u + 210\operatorname{Cos} 2u + 126)$$

&c. &c.

$$\operatorname{Sin} u = \operatorname{Sin} u$$

$$\operatorname{Sin} u^2 = -\tfrac{1}{2} (\operatorname{Cos} 2u - 1)$$

$$\operatorname{Sin} u^3 = -{}^{1} (\operatorname{Sin} 3u - 3\operatorname{Sin} u)$$

$$\operatorname{Sin} u^4 = \tfrac{1}{8} (\operatorname{Cos} 4u - 4\operatorname{Cos} 2u + 3)$$

$$\operatorname{Sin} u^5 = \tfrac{1}{16} (\operatorname{Sin} 5u - 5\operatorname{Sin} 3u + 10\operatorname{Sin} u)$$

$$\operatorname{Sin} u^6 = -\tfrac{1}{32} (\operatorname{Cos} 6u - 6\operatorname{Cos} 4u + 15\operatorname{Cos} 2u - 10)$$

$$\operatorname{Sin} u^7 = -\tfrac{1}{64} (\operatorname{Sin} 7u - 7\operatorname{Sin} 5u + 21\operatorname{Sin} 3u - 35\operatorname{Sin} u)$$

$$\operatorname{Sin} u^8 = \tfrac{1}{128} (\operatorname{Cos} 8u - 8\operatorname{Cos} 6u + 28\operatorname{Cos} 4u - 56\operatorname{Cos} 2u + 35)$$

$$\mathrm{Cos}\,u\,\mathrm{Cos}\,v = \tfrac{1}{2}\left(\mathrm{Cos}\,(u-v) + \mathrm{Cos}\,(u+v)\right)\ ^{*}),$$

atque

$$\mathrm{Sin}\,u^{9} = \tfrac{1}{256}(\mathrm{Sin}\,9u - 9\,\mathrm{Sin}\,7u + 36\,\mathrm{Sin}\,5u - 84\,\mathrm{Sin}\,3u$$
$$+ 126\,\mathrm{Sin}\,u)$$

$$\mathrm{Sin}\,u^{10} = -\tfrac{1}{512}(\mathrm{Cos}\,10u - 10\,\mathrm{Cos}\,8u + 45\,\mathrm{Cos}\,6u - 120\,\mathrm{Cos}\,4u$$
$$+ 210\,\mathrm{Cos}\,2u - 126)$$

&c. &c.

*) Sequuntur hinc quoque formulæ

$$\mathrm{Sin}u\mathrm{Cos}v\mathrm{Sin}x = \tfrac{1}{2}(\mathrm{Sin}(u+v)\mathrm{Sin}x + \mathrm{Sin}(u-v)\mathrm{Sin}x)$$
$$= \tfrac{1}{4}(\mathrm{Cos}(u+v-x) - \mathrm{Cos}(u+v+x) + \mathrm{Cos}(u-v-x)$$
$$- \mathrm{Cos}(u-v+x)),$$

$$\mathrm{Sin}\,u\,\mathrm{Cos}\,v\,\mathrm{Sin}\,x\,\mathrm{Cos}y = \tfrac{1}{4}(\mathrm{Cos}(u+v-x)\mathrm{Cos}y - \mathrm{Cos}(u+v+x).$$
$$\mathrm{Cos}y + \mathrm{Cos}\,(u-v-x)\,\mathrm{Cos}y$$
$$- \mathrm{Cos}(u-v+x)\mathrm{Cos}y)$$
$$= \tfrac{1}{8}(\mathrm{Cos}\,(u+v-x-y) + \mathrm{Cos}(u+v-x+y)$$
$$- \mathrm{Cos}\,(u+v+x-y) - \mathrm{Cos}(u+v+x+y)$$
$$+ \mathrm{Cos}\,(u-v-x-y) + \mathrm{Cos}(u-v-x+y)$$
$$- \mathrm{Cos}(u-v+x-y) - \mathrm{Cos}(u-v+x+y)),$$

sicque porro; quæ transformationi de qua jam agitur quoque inserviunt.

atque

5:o

$$\text{Cos}\, u = \text{Sin}(1^q - u).$$

Memoratas jam formulas trigonometricas idonea adhibendo ratione in seriem formæ indicatæ transformari absque negotio posse functionem nostram $\varkappa$ perspicitur, quod quidem cum effectum fuerit, formam ipsam de qua agitur IV) in adgregatum tantum terminorum formæ

$$A \,.\, \Sigma\, (gx^p + hx^q + \&c.)\, a^x\, \text{Sin}(b + cx),$$

qui cum allata supra II) omnino conveniunt, redactam haberi, manifestum est.

In §:o hujus opellæ V:a observavimus, ad universalem, quantum fieri possit, reddendam formam functionis generalis

$$\Sigma v,$$

adjiciendam illi esse quantitatem quamdam k, quæ vel constans sit arbitraria, vel in genere functio quælibet ipsius x formæ

$$\varphi\ (\text{Sin}\, 2\pi x,\ \text{Cos}\, 2\pi x).$$

Quaeri igitur potest, an, in applicatione formulæ, cui tractandæ hactenus immorati sumus, ipsius b), ubi determinandæ scilicet occurrunt ipsæ

$$\Sigma q_x,\ \Sigma^2 q_x,\ \Sigma^3 q_x,\ \Sigma^4 q_x,\ \&c.,$$

ad